KR-theory

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Complex K-theory

Basic definition

Definition

Let M be a compact manifold.

$$K(M) = \frac{\{[E] - [F] \mid E, F \text{ vector bundles } \}}{\sim}$$

$[E] - [F] \sim [E'] - [F']$ if $\exists G$, $E \oplus F' \oplus G \cong E' \oplus F \oplus G$.

(*Grothendieck group* of vector bundles over M.)

$K(M)$ is an abelian group.
$K(\mathrm{pt}) \cong \mathbb{Z}$ by $[E] - [F] \mapsto \mathrm{rk}(E) - \mathrm{rk}(F)$.
In fact, it is a ring for the product $[E] \cdot [F] = [E \otimes F]$.
Functoriality : if $f : M \to M'$ then $f^* : K(M') \to K(M)$.

K-theory of non compact spaces

First definition : let $\tilde{M}$ be the one-point compactification of M.

$$K(M) = \ker(K(\tilde{M}) \xrightarrow{i^*} K(\infty))$$

where $i : \{\infty\} \to \tilde{M}$ is the inclusion.
Second definition :

$$K(M) = \frac{\{(E, F, d) \mid d : E \to F \text{ iso outside some compact subset}\}}{\sim}$$

where two triples are equivalent if they are stably homotopic.

Basic properties

- Homotopy invariance : if $f_t : X \to Y$ then $f_0^* = f_1^* : K(Y) \to K(X)$.
- Higher K-theory groups : $K^{-n}(X) = K(X \times \mathbb{R}^n)$.
- Bott periodicity : $K^{n+2}(X) \cong K^n(X)$.
- Exactness : if Y is closed in X then

$$\cdots K^{n-1}(X) \to K^{n-1}(Y) \to K^n(X \backslash Y) \to K^n(X) \to K^n(Y) \to \cdots$$

is exact.

Examples

- $K^n(\mathbb{R}^m) = 0$ if $n - m$ odd, $\mathbb{Z}$ otherwise.
- If n even, $K^0(S^n) = \mathbb{Z}^2$, $K^1(S^n) = 0$. If n is odd then $K^0(S^n) = K^1(S^n) = \mathbb{Z}$.

Bott element

Let $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. The nontrivial generator of $[p] = K(S^2)$ is the image of the projection

$$p = \begin{pmatrix} \frac{1+x}{2} & \frac{y+iz}{2} \\ \frac{y-iz}{2} & \frac{1-x}{2} \end{pmatrix}$$

i.e. $E_{(x,y,z)} = p_{x,y,z}(\mathbb{C}^2)$.
$[p] - [1]$ is the image of a generator $\beta \in K(\mathbb{R}^2) \cong \mathbb{Z}$.

Historical motivation : Atiyah-Singer's index theorem

Elliptic operators

Let M be a compact manifold. Let E and F be two vector bundles on M. Let $D : C^\infty(M, E) \to C^\infty(M, F)$ be a differential operator.
Let $\sigma_D : T^*(M) \to \mathcal{L}(E, F)$ be its symbol.
For instance, $\sigma_{a(x)\frac{\partial}{\partial x}}(x, \xi) = a(x)(i\xi)$.
An operator is *elliptic* if its symbol is invertible outside some compact subset of T^*M.
Examples : Laplacian, Dirac operator, signature operator,...

Index problem

Problem : compute

$$\mathrm{Index}_a(D) = \dim(\ker D) - \dim(\mathrm{coker} D) \in \mathbb{Z}.$$

(analytic index).

Atiyah-Singer :

- Use K-theory to define a topological index $\mathrm{Index}_t(D)$;
- Show that both indices coincide ;
- Compute $\mathrm{Index}_t(D)$ in terms of differential forms.

First step : (E, F, σ_D) defines an element of $K(T^*M)$ since D is elliptic.
To obtain a topological index, one needs a map

$$K(T^*M) \to \mathbb{Z}.$$

Elliptic complexes

Let $0 \to E^0 \xrightarrow{d} E^1 \to \cdots \xrightarrow{d} E^n \to 0$ a complex of vector bundles over a (non compact) manifold M such that $d^2 = 0$. It is *elliptic* if for x outside a compact subset of M,

$$0 \to E^0_x \xrightarrow{d} E^1_x \to \cdots \xrightarrow{d} E^n_x \to 0$$

is exact.
Then (E^{even}, E^{odd}, d) determines a K-theory element of M.

Thom isomorphism

Let $\pi : E \to M$ be a complex vector bundle of dimension p. For all $v \in E$, exterior multiplication

$$0 \to \Lambda^0 E_{\pi(v)} \stackrel{v\wedge\cdot}{\to} \Lambda^1 E_{\pi(v)} \stackrel{v\wedge\cdot}{\to} \cdots \stackrel{v\wedge\cdot}{\to} \Lambda^p E_{\pi(v)} \to 0$$

determines an elliptic complex over E, hence an element

$$\lambda_E \in K(E).$$

Then $x \mapsto x \otimes \lambda_E$ is an isomorphism $K(M) \stackrel{\cong}{\to} K(E)$.

Topological index

Let $i : X \to Y$ is an inclusion of manifolds, and N the normal bundle. Let $\pi : TX \to X$.

$$\pi^*(N_{\mathbb{C}}) \cong \text{normal bundle of } (i_* : TX \to TY) \cong U.$$

(U = tubular neighborhood of TX in TY.)

Let $i_! : K(TX) \overset{Thom}{\cong} K(U) \to K(TY)$.

$$\mathrm{Ind}_t : K(TX) \xrightarrow{i_!} K(TE) \xleftarrow{j_!} K(TP) = \mathbb{Z}$$

where $i : X \to E$ embedding in vector space and $j : P \to E$.

Equality of indices

Both Ind_a and Ind_t satisfy :

- $$\begin{array}{ccc} K(TX) & \xrightarrow{i_!} & K(TY) \\ & \overset{Ind}{\searrow} & \downarrow Ind \\ & & \mathbb{Z} \end{array}$$
- If $X = \mathrm{pt}$ then $\mathrm{Ind} = \mathrm{Id}_{\mathbb{Z}}$.

$\Longrightarrow \mathrm{Ind}_a = \mathrm{Ind}_t$.

Why does that help ?

Chern character : if $E \to M$ complex vector bundle, then

$$\mathrm{ch}(E) = [\mathrm{Tr}\ \exp(-\frac{\nabla^2}{2\pi i})] \in H^{even}(M, \mathbb{Q})$$

is a ring homomorphism. It can be extended to a ring isomorphism

$$K^*(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^*(M, \mathbb{Q}).$$

Chern and Thom isomorphism

The Chern character behaves well with respect to the Thom isomorphisms : if $E \to X$ is a complex vector bundle, then

$$\begin{array}{ccc} K(X) & \xrightarrow{Thom} & K(E) \\ \downarrow{ch} & & \downarrow{ch} \\ H^*(X,\mathbb{Q}) & \xrightarrow{Thom} & H^*_c(E,\mathbb{Q}) \end{array}$$

commutes up to multiplication by an element of $H^*(X,\mathbb{Q})$ that can be computed.

Chern and index

It follows that

$$\begin{array}{ccc} K(TM) & \xrightarrow{Ind} & K(P) = \mathbb{Z} \\ \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} \\ H_c^*(TM, \mathbb{Q}) & \xrightarrow{Thom} & H^*(P, \mathbb{Q}) = \mathbb{Q} \end{array}$$

commutes up to multiplication by an element of $H^*(TM, \mathbb{Q})$. In fact,

$$\mathrm{Ind}(D) = (-1)^n \int_{T^*M} \mathrm{ch}(\sigma_D)\mathrm{Td}(T_{\mathbb{C}}M)$$

Examples

- if $\dim(M) = 4k$ is spin then $\hat{A}(M) = \mathrm{ind}(Dirac) \in \mathbb{Z}$
- if M is oriented then $\mathrm{sign}(M) = \mathrm{ind}(d + d^*)$ (index of the signature operator).
- Index theorems for foliations, singular spaces, etc. use K-theory of operator algebras

K-theory has many applications

e.g. in topology : vector field problem on a sphere (Adams, 1962).

If $n = 2^c 16^d u$ (u odd, $0 \leqslant c \leqslant 3$), then the maximal number of linearly independent continuous tangent vector fields on S^{n-1} is $2^c + 8d$.

Some generalizations of complex K-theory

If G is a compact group acting on M, the whole theory is unchanged. Use vector bundles with an action of G.

$$K_G(\mathrm{pt}) = R(G)$$

(ring of representations of G).

If G is not compact, the above definition is not appropriate (KK_G-theory is more adequate).

K-theory of algebras

Let A be a unital algebra.

Definition

An A-module E is finitely generated projective if $\exists E'$, $\exists n$, $E \oplus E' \cong A^n$.

Theorem

(Swan) If M is compact and E $\to$ M is a vector bundle then $C(M, E)$ is a finitely generated projective over $C(M)$.

($\exists E'$, $E \oplus E'$ trivial).
More precisely, there is an equivalence of categories between vector bundles over E and f.g.p. modules over $C(M)$.

Definition

$$K(A) = \frac{\{[E]-[F] \mid E, F \text{ f.g.p. modules over } A\}}{\sim}$$

$[E]-[F] \sim [E']-[F']$ if $\exists G$, $E \oplus F' \oplus G \cong E' \oplus F \oplus G$.

Then $K(M) \cong K(C(M))$.
Remark : one may assume $F = A^n$ for some n.

A f.g.p. is the image of a projection $p \in M_n(A)$ for some n.

Definition

p and p' are similar if $\exists u \in GL(n, A)$, $p' = upu^{-1}$.

Definition

$$K(A) = \frac{\{[p] - [q] \mid p, q \text{ projection in some } M_n(A)\}}{\sim}$$

$[p] - [q] \sim [p'] - [q']$ if $\exists r$, $p \oplus q' \oplus r$ and $p' \oplus q \oplus r$ similar.

This definition agrees with the one using projective modules.

Other point of view

$K^{-n}(A) = \pi_{n-1}(GL_\infty(A))$.
For instance, $K^{-1}(pt)$ corresponds to $\{[E] - n \mid \mathrm{rk}(E) = n\}$ where E is a vector bundle over $S^1 = \mathbb{R}/\mathbb{Z}$. The bundle E is trivial over $[0, 1)$. The identification $E_0 \cong E_1$ comes from an element $u \in GL_n(A)$. Thus, $K^{-1}(A) = \pi_0(GL_\infty(A))$.

KO-theory

Use real vector bundles instead of complex vector bundles.
The theory is 8-periodic.
N.B. *KO*-theory $\neq$ Real *K*-theory.

Example : $K^{-n}(pt) = \mathbb{Z}, 0, \mathbb{Z}, 0, \ldots$
$KO^{-n}(pt) = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \ldots$
For instance, $KO^{-1}(pt) = \pi_0(GL_\infty(\mathbb{R})) = \mathbb{Z}/2$.
$KO^{-2}(pt) = \lim_n \pi_1(SO(n)) = \mathbb{Z}/2$.

Definition

A Real space is a space endowed with an involution $\tau : X \to X$.

We write $\bar{x} = \tau(x)$ for simplicity.
Examples : $\mathbb{R}^{p,q} = \mathbb{R}^q \oplus i\mathbb{R}^p$ with complex conjugation.
$\mathbb{R}^{1,1} \cong \mathbb{C}$.
$P_n(\mathbb{C})$ with complex conjugation.

Definition

Real vector bundle : $E_x \to E_{\bar{x}}$ involution which is antilinear on each fiber.

Real groups

Definition

A Real group is a group which is a Real space such that $\overline{gh} = \bar{g}\bar{h}$.

Actions of Real groups on Real spaces will satisfy $\overline{g \cdot x} = \bar{g} \cdot \bar{x}$ for all $(g, x) \in G \times X$.

One defines KR-theory in the same way.

Definition

Let M be a Real compact manifold.

$$KR(M) = \frac{\{[E]-[F] \mid E, F \text{ Real vector bundles }\}}{\sim}$$

$[E]-[F] \sim [E']-[F']$ if $\exists G$, $E \oplus F' \oplus G \cong E' \oplus F \oplus G$.
(Can add an action of a Real group in the definition to get $KR_G(M)$.)

Real algebras

Definition

A Real algebra is a pair (A, τ) where A is a complex algebra, and $\tau : A \to A$ is an antilinear involution.

e.g. $A = C(M)$ where M is a Real space.

A is $\mathbb{Z}/2\mathbb{Z}$-graded if $A = A^0 \oplus A^1$ with $A^i \cdot A^j \subset A^{i+j}$.
Tensor products of graded algebras : $A \hat{\otimes} B$ is endowed with the product

$$(a \otimes b)(a' \otimes b') = (-1)^{|b|\,|a'|} aa' \otimes bb'.$$

The definition of $KR(A)$ is like the one of $K(A)$ if A is not $\mathbb{Z}/2\mathbb{Z}$-graded.

KR-theory generalizes *KO*-theory

Let $X_{\mathbb{R}} = \{x \in X \mid \bar{x} = x\}$. Restriction of Real vector bundles to $X_{\mathbb{R}}$ are complexifications of real vector bundles over $X_{\mathbb{R}}$, thus there is a map

$$KR(X) \to KR(X_{\mathbb{R}}) \cong KO(X_{\mathbb{R}}).$$

In particular, if τ is trivial then $KR(X) = KO(X_{\mathbb{R}})$.

KR-theory generalizes complex K-theory

Let $S^{p,q}$ be the unit sphere in $\mathbb{R}^{p,q}$.
$S^{1,0}$ consists of two conjugate points.
Let X be a space (without involution).
$X \times S^{1,0}$ consists of two copies of X which are conjugate.

$$KR(X \times S^{1,0}) \cong K(X).$$

Example of *KR*-theory element

Let $X = P(\mathbb{C}^n)$ (the space of complex lines in $\mathbb{C}^n$) with involution=complex conjugation.
The Hopf bundle H is defined by $H_x = x$.
For $n = 2$, $X \cong S^{1,1}$ and $[H] - 1 \in KR(\mathbb{R}^{1,1})$ is a generator.

Basic properties

- Homotopy invariance : if $f_t : X \to Y$ (continuous, involution preserving) then $f_0^* = f_1^* : KR(Y) \to KR(X)$.
- Higher KR-theory groups : $KR^{-p,-q}(X) = KR(X \times \mathbb{R}^{p,q})$.
- $KR^{p+r,q+r}(X) \cong KR^{p,q}(X)$. Denote this group by $KR^{p-q}(X)$.
- Bott periodicity : $KR^{n+8}(X) \cong KR^n(X)$.
- Exactness : if Y is closed in X then

$$\cdots KR^{n-1}(X) \to KR^{n-1}(Y) \to KR^n(X \setminus Y)$$

$$\to KR^n(X) \to KR^n(Y) \to \cdots$$

is exact.

External product in Real K-theory

For X, Y compact,

$$KR(X) \otimes KR(Y) \to KR(X \times Y)$$

is defined by $[E] \otimes [F] \mapsto [E \times F]$.
This extends to

$$KR^{p,q}(X) \otimes KR^{p',q'}(Y) \to KR^{p+p',q+q'}(X \times Y).$$

This is still valid for X, Y noncompact (recall $KR(X) \subset KR(\tilde{X})$).

Description of $KR(pt)$

Recall $K^{-n}(pt) = \mathbb{Z}$, 0, $\mathbb{Z}$, 0, $\dots$
Denote by $\mu_2 = [H] - [1]$ the Bott generator, then μ_2^k generates $K^{2k}(pt)$.

$KR^{-n}(pt) = KO^{-n}(pt) = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \dots$.
Let $\eta_1 = [L] - [1]$ where $[L]$ is the Möbius band. Then η_1 and η_1^2 are $\neq 0$.
$\eta_4^2 = 4$.

The complexification operation

$$c : KR(X) \to K(X)$$

is the forgetful functor (forget the involution).

Particular case : if $X = X_{\mathbb{R}}$ then

$$c : KO(X) \to K(X), \qquad [E] \mapsto [E \otimes_{\mathbb{R}} \mathbb{C}].$$

The realization operation

$$\begin{array}{rcl} r\colon K(X) & \to & KR(X) \\ [E] & \mapsto & [E \otimes \tau^* \bar{E}] \end{array}$$

where $(\tau^* \bar{E})_x = \overline{E_{\bar{x}}}$.

The Real structure on $E \otimes \tau^* \bar{E}$ is $\tau(\xi_x, \overline{\eta_{\bar{x}}}) = (\eta_{\bar{x}}, \overline{\xi_x})$.

$$cr(\alpha) = \alpha + \alpha^*$$

where $* : K(X) \to K(X)$, $[E] \mapsto [\tau^* \bar{E}]$.

$$rc(\alpha) = 2\alpha$$

because of the isomorphism

$$\begin{array}{ccc} E_x \oplus E_x & \to & E_x \oplus \overline{E_{\bar{x}}} \\ (\xi, \eta) & \mapsto & (\xi + i\eta, \bar{\xi} - i\bar{\eta}). \end{array}$$

Case $X = X_{\mathbb{R}}$

$$KO(X) \underset{r}{\overset{c}{\rightleftarrows}} K(X).$$

$$c(KO(X)) \subset \{\alpha \in K(X) \mid \alpha^* = \alpha\} =: K(X)^0.$$

$$KO(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \cong K(X)^0 \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}].$$

Case $X = pt$

$c([1]) = 1$, $c(\eta_1) = c(\eta_1^2) = 0$, $c(\eta_4) = 2\mu_2^2$,
$r([1]) = [2]$, $r(\mu_2) = \eta_1^2$, $r(\mu_2^2) = \eta_4$.

Clifford algebras

Let Q be a quadratic form on an $\mathbb{R}$-vector space V. The Clifford algebra of (V, Q) is the quotient of the free algebra over V by the relations

$$v \cdot v = Q(v).$$

For instance, if $Q = 0$ then $\mathrm{Cliff}(V, Q) = \Lambda V$ has dimension 2^n.
If V has a Q-orthogonal basis $(e_1, \ldots, e_n)$, then $\mathrm{Cliff}(V, Q)$ has basis

$$(e_{i_1} e_{i_2} \cdots e_{i_k})_{1 \leqslant i_1 < i_1 < \cdots i_k \leqslant n}.$$

Products can be computed by $e_i^2 = Q(e_i)\mathbf{1}$ and $e_i e_j + e_j e_i = 0$ if $i \neq j$.
$\dim(\mathrm{Cliff}(V, Q)) = 2^n$ but $\mathrm{Cliff}(V, Q) \not\cong \Lambda V$.
$\mathrm{Cliff}(V, Q)$ is $\mathbb{Z}/2\mathbb{Z}$-graded.

Clifford algebras

Consider
$Q(x_1,\dots,x_q,x_{q+1},\dots,x_{q+p}) = x_1^2+\cdots+x_q^2-(x_{q+1}^2+\cdots+x_{q+p}^2)$
the quadratic form of signature (q,p) on $\mathbb{R}^{p+q}$.
Denote by $C^{p,q}$ its Clifford algebra.
$C^{p,0} = \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{H}\oplus\mathbb{H},$
$M_2(\mathbb{H}), M_4(\mathbb{C}), M_8(\mathbb{R}), M_8(\mathbb{R})\oplus M_8(\mathbb{R}), M_{16}(\mathbb{R}),\dots$
$C^{0,q} = \mathbb{R},\mathbb{R}\oplus\mathbb{R}, M_2(\mathbb{R}), M_2(\mathbb{C}), M_2(\mathbb{H}),$
$M_2(\mathbb{H})\oplus M_2(\mathbb{H}), M_4(\mathbb{H}), M_8(\mathbb{C}), M_{16}(\mathbb{R}),\dots$.

Using $C^{p,q}\hat{\otimes}C^{p',q'} \cong C^{p+p',q+q'}$, one can compute $C^{p,q}$ for all p,q.

Let $A = A^0 \oplus A^1$ be a Real $\mathbb{Z}/2\mathbb{Z}$-graded algebra.
A Real, $\mathbb{Z}/2\mathbb{Z}$-graded module over A is an A module of the form $E = E^0 \oplus E^1$, endowed with an antilinear involution τ, such that

- $E^i \cdot A^j \subseteq E^{i+j}$;
- $\tau(\xi a) = \tau(\xi)\tau(a)$ for all $(\xi, a) \in E \times A$.

K-theory of graded, unital algebras

The algebra $\mathrm{End}(E)$ is again $\mathbb{Z}/2$-graded and Real :

- $\deg(T) = 1$ if $\deg(T(\xi)) = \deg(\xi) + 1 \pmod 2$ for all $\xi \in E$;
- $\bar{T}(\xi) = \overline{T(\bar{\xi})}$.

The (graded) KR-theory of A is the set of graded, Real A-modules modulo degenerate ones.

Degenerate means : $\exists T \in \mathrm{End}_A(E)$, $T^2 = \mathrm{Id}$, $\deg(T) = 1$, $T = \bar{T}$.

This is coherent with the def for ungraded algebras

$$[E] = [E^0 \oplus E^1] \text{ graded} \longrightarrow [E^0] - [E^1].$$

$KO(C^{0,1}) = 0$

$C^{0,1} = \langle 1, \varepsilon \rangle$ with $\varepsilon^2 = 1$, ϵ of degree 1.
If $E = E^0 \oplus E^1$ is a graded $C^{0,1}$-module, then $\varepsilon : E^0 \xrightarrow{\cong} E^1$. We can assume $E = E^0 \oplus E^0$ and

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Take $T = \varepsilon \implies E$ is degenerate.

$KO(C^{1,0}) = \mathbb{Z}/2\mathbb{Z}$

$C^{1,0} = \langle 1, e \rangle$ with $e^2 = -1$, e of degree 1.
If $E = E^0 \oplus E^1$ is a graded $C^{1,0}$-module, then $e : E^0 \overset{\cong}{\to} E^1$. We can assume $E = E^0 \oplus E^0$ and

$$e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

If $T = \begin{pmatrix} 0 & S^{-1} \\ S & 0 \end{pmatrix}$ is an endomorphism of E, it commutes with e so $S^2 = -1$. Possible iff $\dim(E^0)$ is even.

$C^{2,0} = \langle 1, e_1, e_1, e_1 e_2 \rangle$ with $e_1^2 = e_2^2 = -1$, $e_1 e_2 = -e_2 e_1$, e_j of degree 1, i.e. $C^{2,0} \cong \mathbb{H}$.
Identify $\mathbb{C}$ with $\langle 1, e_1 e_2 \rangle$.
If $E = E^0 \oplus E^1$ is a graded $C^{2,0}$-module, then E^j are $\mathbb{C}$-vector spaces and $e_1 : E^0 \overset{\cong}{\to} E^1$. We can assume $E = E^0 \oplus E^0$ and

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

If $T = \begin{pmatrix} 0 & S^{-1} \\ S & 0 \end{pmatrix}$ is an endomorphism of E, then S is antilinear and $S^2 = -\mathrm{Id}$. Possible iff $\dim(E^0)$ is even.

$KO(C^{4,0}) = \mathbb{Z}$

$$C^{4,0} \cong C^{0,4} \cong M_2(\mathbb{H})$$

graded by $\deg \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} = 0$, $\deg \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = 1$.

In general $K(M_n(A)) \cong K(A)$, so

$$KO(C^{4,0}) = KO(C^{0,4}) = KO(\mathbb{H}) = \mathbb{Z}.$$

The generator is $[\mathbb{H}]$ ($\mathbb{H}$ is a module over itself).

$KO(C^{p,p}) = \mathbb{Z}$

$$\mathrm{Cliff}(V)\hat{\otimes}\mathrm{Cliff}(-V) \cong \mathcal{L}(\Lambda^* V).$$

Thus $C^{p,p} \cong M_{2p}(\mathbb{R})$.

$$KR^{p,q}(A) \cong KR(A \hat{\otimes} C^{p,q}).$$

$\eta_1 = [C^{1,0}] \in KO(C^{1,0}) \cong KR^{1,0}(pt)$ ($C^{1,0}$ is a module over itself).
$\eta_2 = [C^{2,0}] \in KO(C^{2,0}) \cong KR^{2,0}(pt)$.
$\eta_4 = [\mathbb{H}^2] \in KO(M_2(\mathbb{H})) \cong KO(C^{4,0}) \cong KR^{4,0}(pt)$.
If this is granted, then $KR^{p,q}(A) \cong KR^{p+r,q+r}(A) = K^{p-q}(A)$ and $KR^{p+8}(A) \cong KR^{p}(A)$.

Check that $c(\eta_4) = 2\mu_2^2$

$\eta_4 = [\mathbb{H}]$ as an $\mathbb{H}$-module.
$\mathbb{H} \otimes \mathbb{C} \cong M_2(\mathbb{C})$ because

$$\mathbb{H} = \{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C} \},$$

and $M_2(\mathbb{C}) = \mathbb{H} \oplus i\mathbb{H}$.

As a $M_2(\mathbb{C})$-module, $[M_2(\mathbb{C})] = 2[\mathbb{C}^2]$.

Check that $\eta_4^2 = 4$

$$\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R}),$$

which is isomorphic to $4[\mathbb{R}^4]$ as a $M_4(\mathbb{R})$-module.

$KR^{n-4}(X)$ and $KR^n(X)$

Composition of multiplications by η_4

$$KR^n(X) \to KR^{n-4}(X) \to KR^{n-8}(X) \cong KR^n(X)$$

is 4 Id, thus

$$KR^n(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \cong KR^{n-4}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}].$$

Remark : $KR^{-4}(X) = KR(C(X, \mathbb{H})_{\mathbb{C}})$ is quaternionic K-theory. A quaternionic vector bundle on (X, τ) is a complex vector bundle $E \to X$ together with a conjugate linear map $J : E \to E$ commuting with the Real structure on X, such that $J^2 = -\mathrm{Id}$.

This group cannot always be identified with a “known” K-theory group but can be computed by means of a Gysin-type exact sequence.

$$\cdots \to KR^{p-q}(X) \stackrel{\eta_1^p \cdot}{\to} KR^{-q}(X) \stackrel{p_1^*}{\to} KR^{-q}(X \times S^{p,0}) \to \cdots$$

$KR(X \times S^{p,0})$

Recall $KR(X \times S^{1,0}) \cong K(X)$.
$KR(X \times S^{2,0}) = KSC(X)$ (self-conjugate K-theory) is the Grothendieck group of homotopy classes of self-conjugate bundles over X.

(A self-conjugate bundle over (X, τ) is a complex vector bundle $E \to X$ together with an isomorphism $\alpha\colon E \to \overline{\tau^* E}$.)

For $p \geqslant 3$, using $\eta_1^3 = 0$ we have an exact sequence

$$0 \to KR^{-q}(X) \xrightarrow{p_1^*} KR^{-q}(X \times S^{p,0}) \to KR^{p+1-q}(X) \to 0$$

Remark : index theory for families

If $\pi : Y \to X$ is a Real fibration, $E \to Y$ a Real vector bundle and $D = (D_x)_{x \in X}$ is a family of Real, elliptic differential operators, one can construct an index

$$\mathrm{Ind}(D) \in KR(X).$$

K-theory and Fredholm operators

Elliptic operators on a manifold provide Fredholm operators on a Hilbert space. Hence, K-theory elements are often conveniently described in terms of Fredholm operators.

(T is Fredholm $\iff \exists S$, $TS - \mathrm{Id}$ and $ST - \mathrm{Id}$ are of finite rank, $\iff \exists S'$ TS' and $S'T$ are compact, $\iff$ ker T and coker(T) are finite dimensional.)
Example : $\pi_0(\text{Fred}) \cong K(pt)$ given by $T \mapsto [\ker T] - [\mathrm{coker}(T)]$.
The composition

$$\text{Fred} \to K(pt) \overset{\cong}{\to} \mathbb{Z}$$

is $T \mapsto \mathrm{ind}(T)$.

K-theory and Fredholm operators

More generally, one can show that for X compact, $K(X) = [X, \mathrm{Fred}]$.

$K(X)$ is the Grothendieck group of homotopy classes of families $D = (D_x)_{x \in X}$ of self-adjoint, degree 1 Fredholm operators.

$$D_x = \begin{pmatrix} 0 & T_x^* \\ T_x & 0 \end{pmatrix}.$$

(If X is locally compact and not compact, one needs to add a condition of vanishing at infinity.)

Advantages of this definition

- No need of one-point compactifications
- Easily extends to ($\mathbb{Z}/2\mathbb{Z}$-graded) Clifford bundles
- The external product $K(X) \times K(Y) \to K(X \times Y)$ is described by $[D] \cdot [D'] = [D \hat{\otimes} 1 + 1 \hat{\otimes} D']$.

The Bott element

Let (V, Q) be a (real) quadratic space. Let $A = C_0(V, \mathrm{Cliff}(V, Q))$. Then

$$KO(A) = \langle [\beta] \rangle \cong \mathbb{Z}$$

where $[\beta]$ is the "Bott element" :

$$\beta_v = \text{Clifford multiplication by } \frac{v}{\sqrt{1 + ||v||^2}}.$$

Note that β is indeed of degree 1, that $1 - \beta_v^2 = \dfrac{1}{1 + ||v||^2}$ tends to 0 as $||v|| \to \infty$.

For $V = \mathbb{R}^{p,q}$, this gives

$$KR(X) \cong KR(C_0(X \times \mathbb{R}^{p,q}, C^{p,q})_{\mathbb{C}}).$$

More generally, if $\pi : V \to X$ is a Real vector bundle, there is a Thom isomorphism

$$KR^n(X) \cong KR^n(C_0(V, \pi^*\mathrm{Cliff}(V)_{\mathbb{C}})).$$

If there exists a spin structure on V, then $KR^n(X) \cong KR^n(V)$. Otherwise, $KR(X)$ is a twisted K-theory group of V.

A few references

- (On complex K-theory and index theory) Atiyah and Singer, *The index of elliptic operators* I, II, III,...
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- (For operator algebraists :) Moutuou, PhD thesis.