Bulk-edge duality for topological insulators

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Journes de Physique Mathmathique Lyon
Topological Insulators
September 11-13, 2013

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joint work with Marcello Porta
thanks to Yosi Avron

Introduction

Rueda de casino

Hamiltonians

Indices

Topological insulators: first impressions

- Insulator in the Bulk: Excitation gap
  For independent electrons: band gap at Fermi energy

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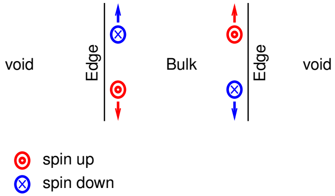

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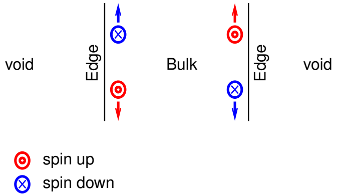

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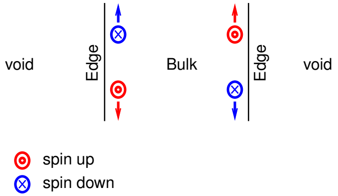

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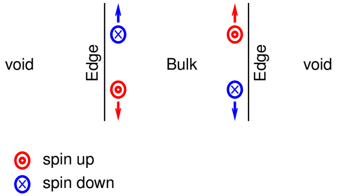

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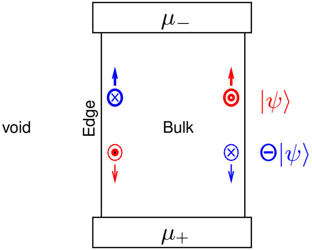
Contributors to the field: Kane, Mele, Zhang, Moore; Fröhlich

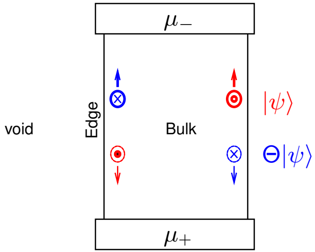
A technological application



$\mu_+ > \mu_-$

Θ time-reversal

A technological application



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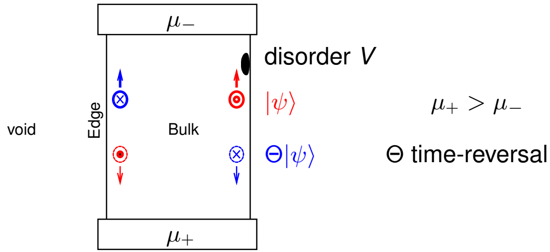
Θ time-reversal

- On the two edges (net): parallel charge currents, anti-parallel spin currents

A technological application

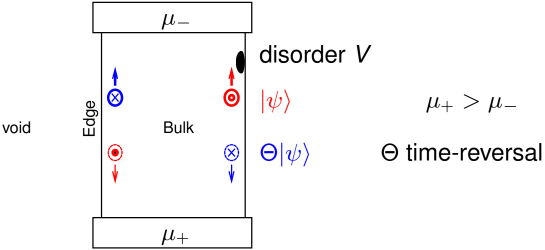


- On the two edges (net): parallel charge currents, anti-parallel spin currents
- Stable against backscattering, $|\psi\rangle \to \Theta|\psi\rangle$, induced by disorder

$$\langle \Theta\psi | V | \psi \rangle = 0$$

if V is time-reversal invariant.

A technological application



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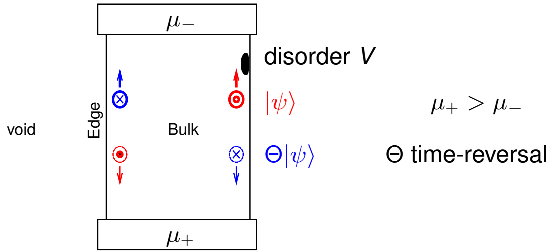

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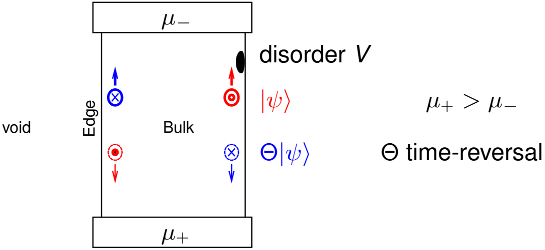

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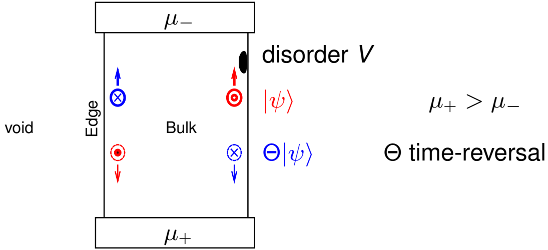

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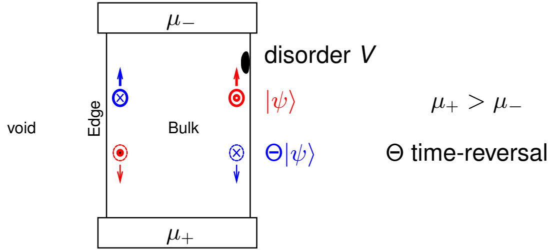

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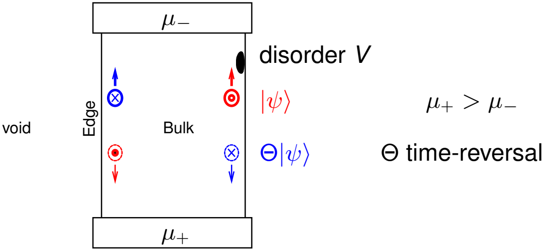
A technological application



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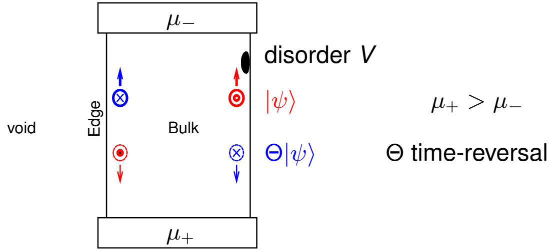

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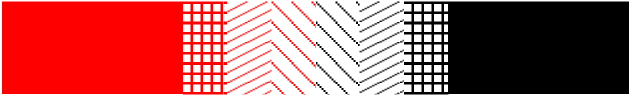
Bulk-edge correspondence

Deformation as interpolation in physical space:



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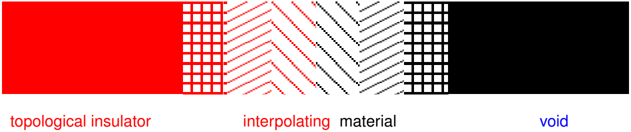

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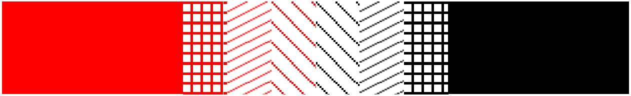

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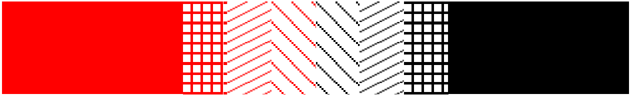

topological insulator interpolating material void

- Gap must close somewhere in between. Hence: Interface states at Fermi energy.
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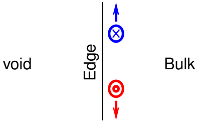
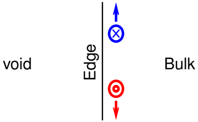
- Bulk-edge correspondence: Termination of bulk of a topological insulator implies edge states. (But not conversely!)

Bulk-edge correspondence



In a nutshell: Termination of bulk of a topological insulator implies edge states

Bulk-edge correspondence

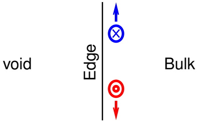


In a nutshell: Termination of bulk of a topological insulator implies edge states

- State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

Bulk-edge correspondence



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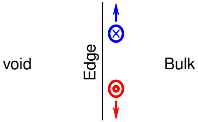

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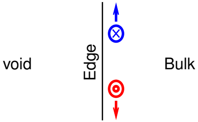
Bulk-edge correspondence. Done?



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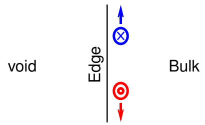
- State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

- Express that property as an Index relating to the Bulk, resp. to the Edge. Yes, e.g. Kane and Mele.
- Bulk-edge duality: Can it be shown that the two indices agree? Schulz-Baldes et al.; Essin & Gurarie

Bulk-edge correspondence. Today



In a nutshell: Termination of bulk of a topological insulator implies edge states

- State the (intrinsic) topological property distinguishing different classes of insulators.

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Introduction

Rueda de casino

Hamiltonians

Indices

Rueda de casino. Time $0'15''$

Rueda de casino. Time $0'44''$

Rueda de casino. Time $0'44.25''$

Rueda de casino. Time $0'44.50''$

Rueda de casino. Time $0'44.75''$

Rueda de casino. Time $0'45''$

Rueda de casino. Time $0'45.25''$

Rueda de casino. Time $0'45.50''$

Rueda de casino. Time $3'23''$

Rules of the dance

Dancers

- start in pairs, anywhere
- end in pairs, anywhere (possibly elseways & elsewhere)
- are free in between
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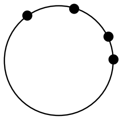
There are dances which can not be deformed into one another.

Which is the index that makes the difference?

The index of a Rueda

A snapshot of the dance



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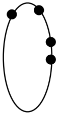
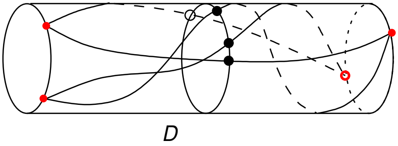


Dance D as a whole



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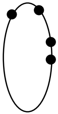

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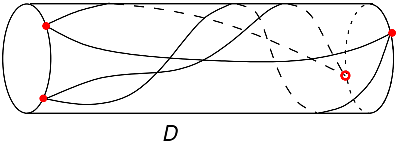

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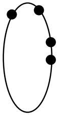

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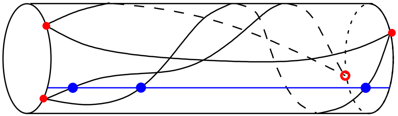

D

The index of a Rueda

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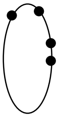

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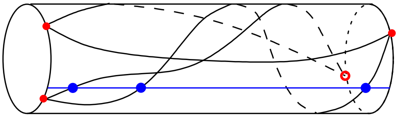


D

$\mathcal{I}(D) =$ parity of number of crossings of fiducial line

Introduction

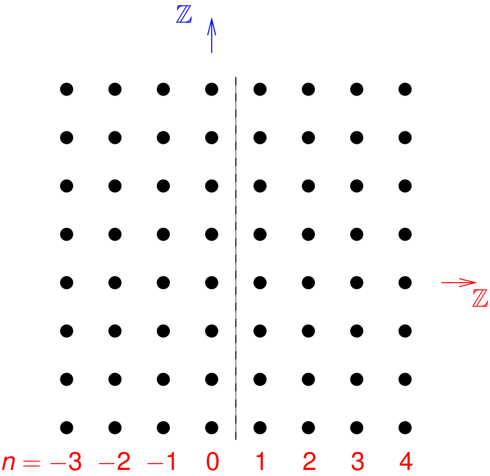
Rueda de casino

Hamiltonians

Indices

Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)



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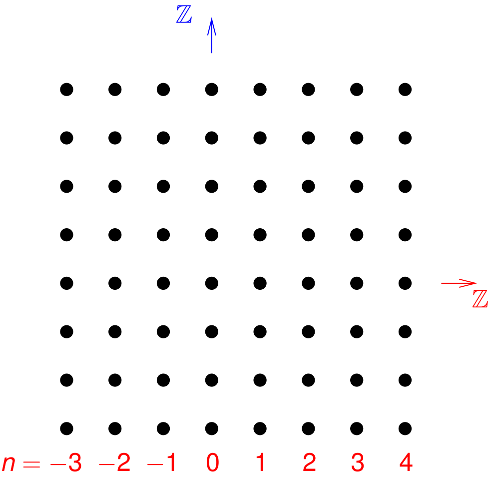

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End up with wave-functions $\psi = (\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C}^N)$ and Bulk Hamiltonian

$$\big(H(k)\psi\big)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n(k)\psi_n$$

with
$V_n(k) = V_n(k)^* \in M_N(\mathbb{C})$ (potential)
$A(k) \in \mathrm{GL}(N)$ (hopping)

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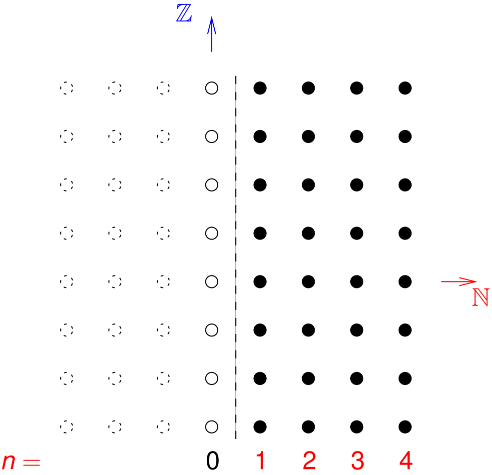
Edge Hamiltonian

Hamiltonian on the lattice $\mathbb{N} \times \mathbb{Z}$ (half-plane) with $\mathbb{N} = \{1, 2, \ldots\}$



Edge Hamiltonian

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- translation invariant as before (hence Bloch reduction)

Wave-functions $\psi \in \ell^2(\mathbb{N}; \mathbb{C}^N)$ and Edge Hamiltonian

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- has Dirichlet boundary conditions: for $n = 1$ set $\psi_0 = 0$

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- translation invariant as before (hence Bloch reduction)

Wave-functions $\psi \in \ell^2(\mathbb{N}; \mathbb{C}^N)$ and Edge Hamiltonian

$$\left(H^\sharp(k)\psi\right)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n^\sharp(k)\psi_n$$

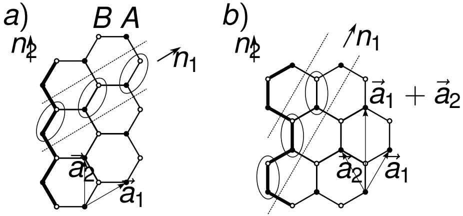
which

- agrees with Bulk Hamiltonian outside of collar near edge (width n_0)
$$V_n^\sharp(k) = V_n(k) \, , \qquad (n > n_0)$$
- has Dirichlet boundary conditions: for $n = 1$ set $\psi_0 = 0$

Note: $\sigma_{\text{ess}}(H^\sharp(k)) \subset \sigma_{\text{ess}}(H(k))$, but typically $\sigma_{\text{disc}}(H^\sharp(k)) \not\subset \sigma_{\text{disc}}(H(k))$

Graphene as an example

Hamiltonian is nearest neighbor hopping on honeycomb lattice



(a) zigzag, resp. (b) armchair boundaries

Dimers ($N = 2$).

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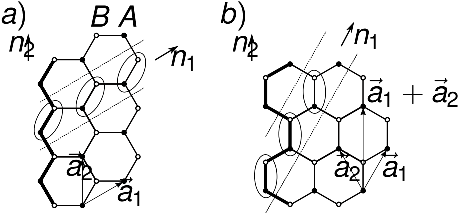

(a) zigzag, resp. (b) armchair boundaries

Dimers ($N = 2$). For (b):

$$\psi_n = \begin{pmatrix} \psi_n^A \\ \psi_n^B \end{pmatrix} \in \mathbb{C}^{N=2} , \quad A(k) = -t \begin{pmatrix} 0 & 1 \\ \mathrm{e}^{\mathrm{i}k} & 0 \end{pmatrix} , \quad V_n(k) = -t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For (a): too, but $A(k) \notin \mathrm{GL}(N)$

Also: Extensions with spin, spin orbit coupling leading to topological insulators (Kane & Mele)

General assumptions

- Gap assumption: Fermi energy μ lies in a gap for all $k \in S^1$:

$$\mu \notin \sigma(H(k))$$

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- Fermionic time-reversal symmetry: $\Theta : \mathbb{C}^N \to \mathbb{C}^N$
  - Θ is anti-unitary and $\Theta^2 = -1$;
  - For all $k \in S^1$,
  $$H(-k) = \Theta H(k) \Theta^{-1}$$
  where Θ also denotes the map induced on $\ell^2(\mathbb{Z}; \mathbb{C}^N)$. Likewise for $H^\sharp(k)$

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$$H\psi = E\psi \implies H(\Theta\psi) = E(\Theta\psi)$$

and $\Theta\psi = \lambda\psi$, $(\lambda \in \mathbb{C})$ is impossible:

$$-\psi = \Theta^2\psi = \bar{\lambda}\Theta\psi = \bar{\lambda}\lambda\psi \qquad (\Rightarrow\Leftarrow)$$

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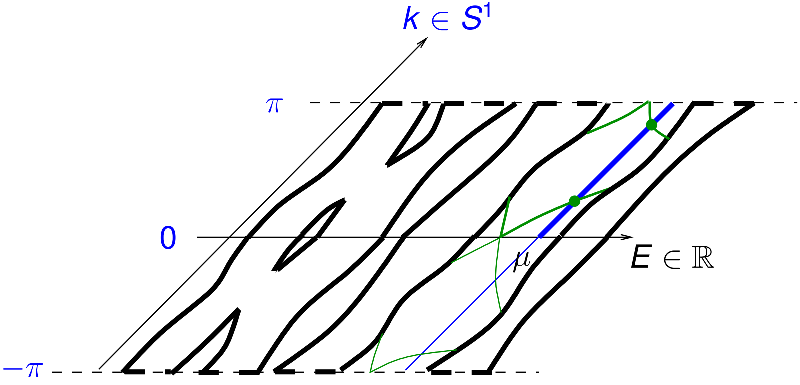


Bands, Fermi line (one half **fat**), edge states

Introduction

Rueda de casino

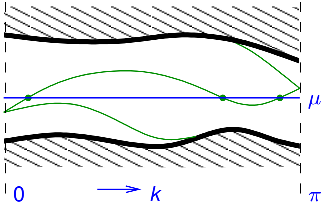
Hamiltonians

Indices

The edge index

The spectrum of $H^\sharp(k)$

symmetric on $-\pi \leq k \leq 0$



Bands, Fermi line, edge states

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$$\mathcal{I}^\sharp = \text{parity of number of eigenvalue crossings}$$

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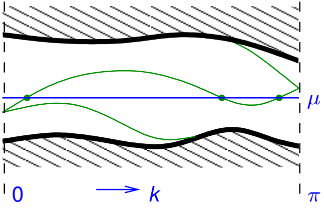

Bands, Fermi line, edge states

Definition: Edge Index

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At fixed k, map gap to $S^1 \setminus \{1\}$ and bands to $1 \in S^1$:
Edge Index is index of a rueda.

Towards the bulk index

Let $z \in \mathbb{C}$. The Schrödinger equation

$$(H(k) - z)\psi = 0$$

(as a 2nd order difference equation) has $2N$ solutions $\psi = (\psi_n)_{n \in \mathbb{Z}}$, $\psi_n \in \mathbb{C}^N$.

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Let $z \notin \sigma(H(k))$. Then

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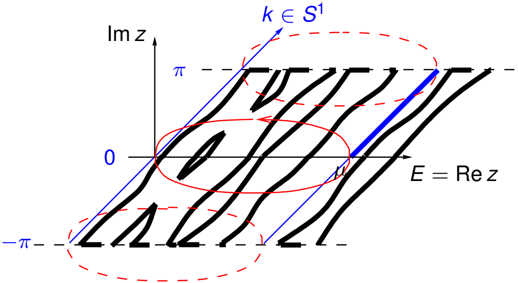
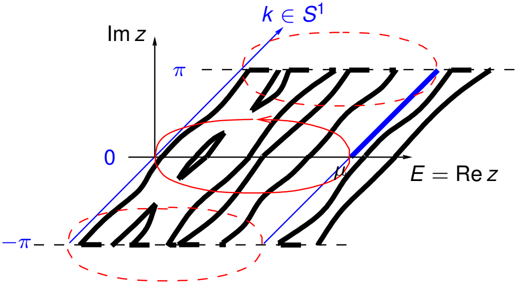


Loop γ and torus $\mathbb{T} = \gamma \times S^1$

Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$, and involution Θ.

The bulk index



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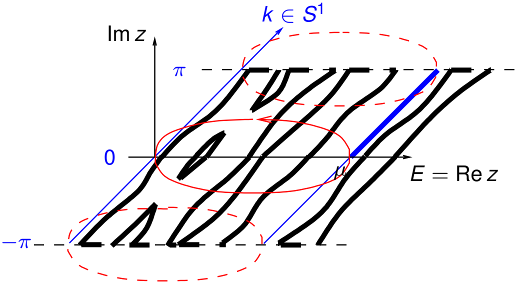

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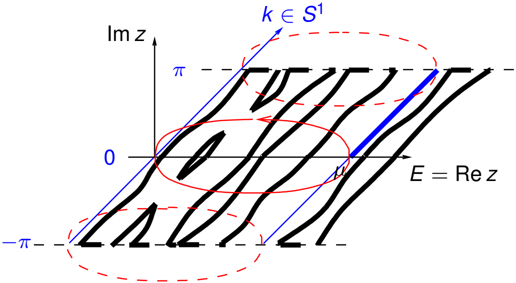

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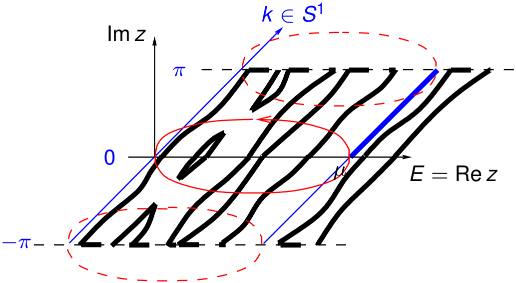

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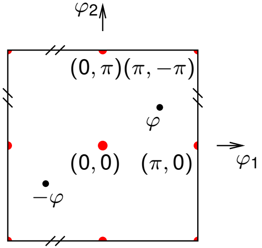
Time reversal invariant bundles $(E, \mathbb{T}, \Theta)$

$\mathbb{T}$



- $\mathbb{T} \ni \varphi = (\varphi_1, \varphi_2)$

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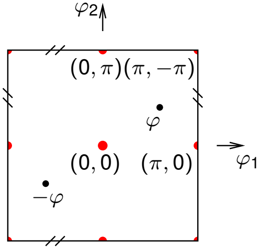

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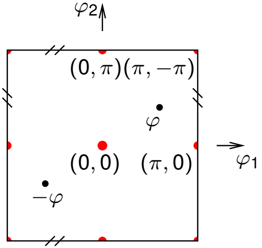

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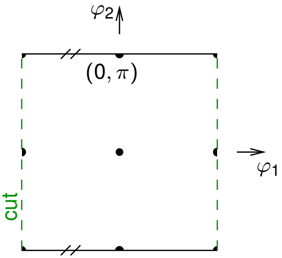
- Frame bundle $F(E)$ has fibers $F(E)_\varphi \ni v = (v_1, \ldots v_N)$ consisting of bases v of E_φ.

Classification of time reversal invariant bundles

Consider the cut torus:



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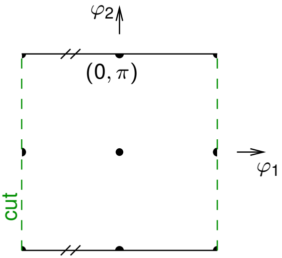

Lemma On the cut torus the frame bundle admits a section $\varphi \mapsto v(\varphi) \in F(E)_\varphi$ which is time-reversal invariant:

$$v(-\varphi) = (\Theta v(\varphi))\varepsilon$$

with ε the block diagonal matrix with blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Idea: At a time reversal invariant point, that means ($N = 2$)

$$v_2 = \Theta v_1 \qquad v_1 = -\Theta v_2$$

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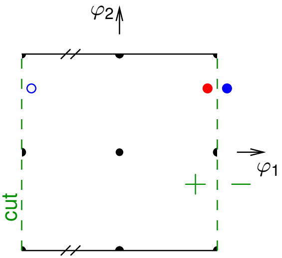

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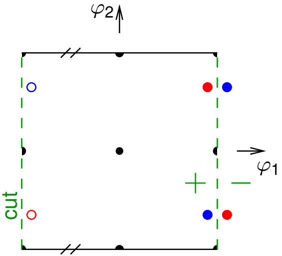

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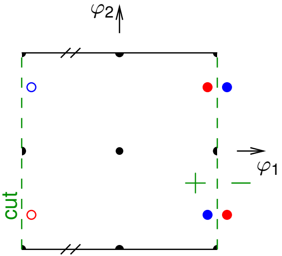

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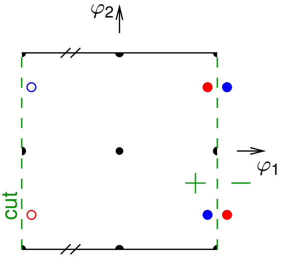

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The statement of the theorem is now complete.

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...aside ends here.

Main result

Theorem Bulk and edge indices agree:

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$\mathcal{I} = +1$: ordinary insulator
$\mathcal{I} = -1$: topological insulator

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- For this slide only: $N = 1$.

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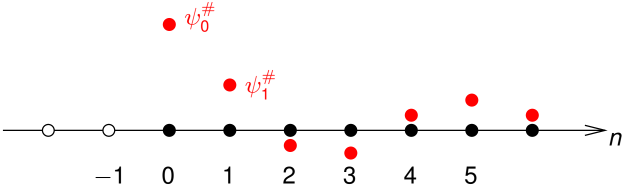

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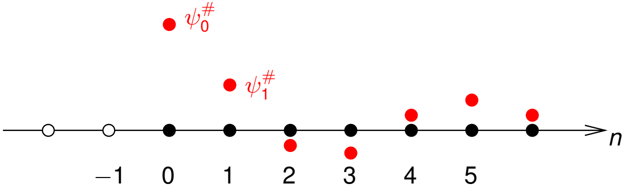

- Solution is unique up to multiples

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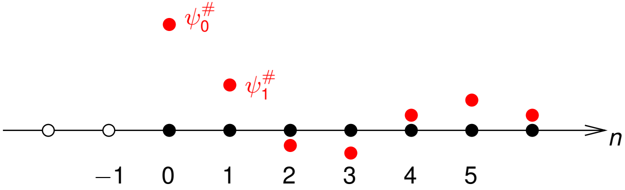

- Solution is unique up to multiples
- $\psi_0^\sharp = 1$ picks a unique solution,

Proof of Theorem (preliminary remark)

- For this slide only: $N = 1$. Schrödinger (2nd order difference) equation on the half-line

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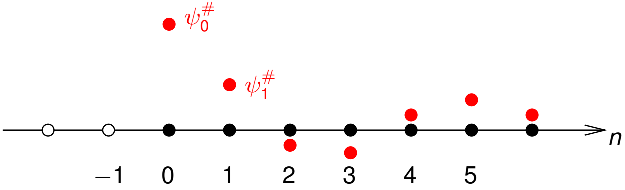
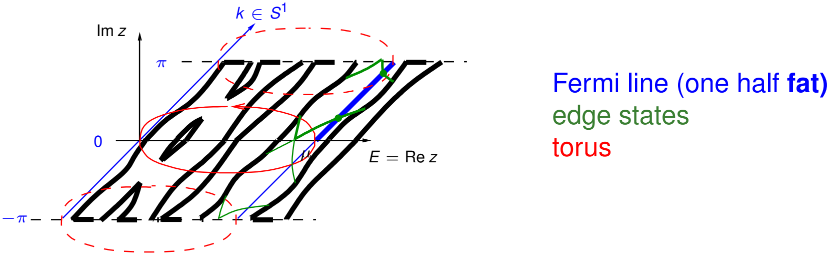
with solution $\psi_n^\sharp \in \mathbb{C}$, $(n = 0, 1, 2)$ decaying at $n \to \infty$



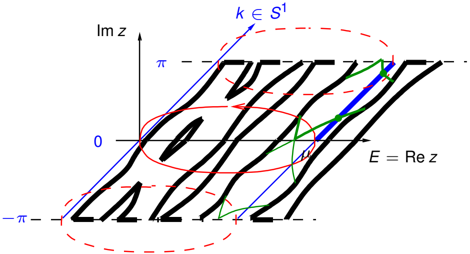
- Solution is unique up to multiples
- $\psi_0^\sharp = 1$ picks a unique solution, except if $n = 0$ is a node

Proof of Theorem (sketch)



- ψ, $\psi^\sharp$ solutions (bulk, edge) at z, k decaying at $n \to +\infty$
- Bijective map $\psi \mapsto \psi^\sharp$, so that $\psi_n = \psi^\sharp_n$ ($n > n_0$)
- $\exists \psi^\sharp \neq 0 \mid \psi^\sharp_{n=0} = 0 \Leftrightarrow z \in \sigma(H^\sharp(k))$
- There is a section of the frame bundle $F(E)$, global on $\mathbb{T}$, except at edge eigenvalue crossings
- Cut the torus along the Fermi line; let $T(k)$ be the transition matrix
- There $T(k) = \mathbb{I}_N$, except near eigenvalue crossings
- As k traverses one of them, $T(k)$ has eigenvalues 1 (multiplicity $N-1$) and $\lambda(k)$ making one turn of S^1

Proof of Theorem (sketch)



Fermi line (one half **fat)**
edge states
torus

- $\psi, \psi^\sharp$ solutions (bulk, edge) at z, k decaying at $n \to +\infty$
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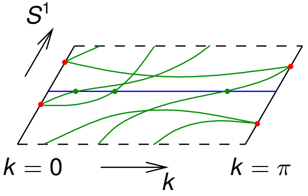
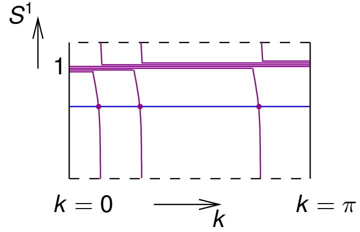
Proof of Theorem: Dual ruedas

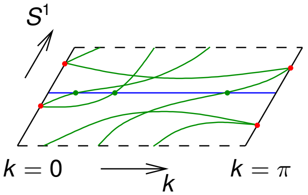


Edge rueda: edge eigenvalues

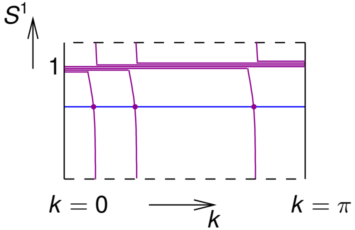


Bulk rueda: eigenvalues of $T(k)$

Proof of Theorem: Dual ruedas



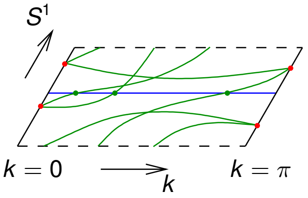
Edge rueda: edge eigenvalues



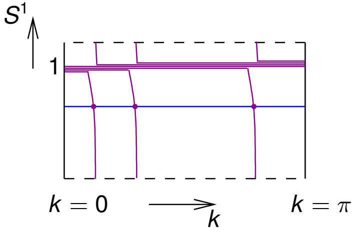
Bulk rueda: eigenvalues of $T(k)$

Ruedas share intersection points.

Proof of Theorem: Dual ruedas



Edge rueda: edge eigenvalues



Bulk rueda: eigenvalues of $T(k)$

Ruedas share intersection points. Hence indices are equal □

Final remarks

Further results:

- In case the Bulk Hamiltonian is doubly periodic:

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with product over filled pairs

Final remarks

Further results:

- In case the Bulk Hamiltonian is doubly periodic: Brillouin zone serves as torus and (j-th pair of) Bloch solutions as bundles E_j. Then
$$\mathcal{I} = \prod_j \mathcal{I}(E_j)$$
with product over filled pairs
- A direct link between indices of Bloch bundles and the edge index via Levinson's theorem.
- 3d topological insulators (weak and strong indices: 3+1)

Open questions:

- No periodicity (disordered case)?

Summary

Bulk = Edge

$$\mathcal{I} = \mathcal{I}^\sharp$$